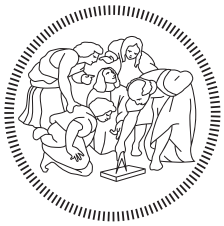

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School of Industrial and Information Engineering

Materials Engineering and Nanotechnology

Mathematical methods for materials engineering

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Author's preface

Dear Reader,
I am delighted to present this comprehensive collection of lecture notes for **'Mathematical methods for materials engineering'**. These notes have been thoughtfully structured to closely align with the course curriculum, providing an invaluable resource for your studies.

These lecture notes are a culmination of extensive research and diligent note-taking, incorporating insights from professor lectures, course materials, textbooks, and other reputable sources. By bringing together these diverse resources, I aimed to provide you with a well-rounded understanding of the subject matter.

While every effort has been made to ensure accuracy and clarity, it is essential to acknowledge that errors or discrepancies may exist within these notes. The complexities inherent in the subject matter, coupled with the limitations of human interpretation, make occasional inaccuracies unavoidable. Therefore, I encourage you to approach these notes critically, supplementing your understanding with additional academic sources and seeking clarification from your professors or peers when necessary.

I have written these lecture notes using LaTeX, a precise typesetting system widely used in scientific and academic writing. Its utilization ensures a visually appealing and organized document, enhancing the overall readability and accessibility of the content.

I sincerely hope that these lecture notes will serve as a valuable companion throughout your academic journey. They are designed to supplement your learning experience, providing a comprehensive overview of the course material. Remember to approach these notes as a guide, actively engaging in discussions, seeking further insights and embracing the collaborative spirit of academia.

Wishing you success and an enriching learning experience.

Milano, Italy
February 6, 2026

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Part I

Functional Analysis and Preliminary Tools

1 Introduction to Function Spaces

In mathematical analysis, and particularly in functional analysis and distribution theory, it is often necessary to study classes of functions with specific properties of smoothness, decay, or support.
These spaces provide the framework in which tools such as the Fourier transform, distributions, and generalized functions can be rigorously defined.

> A central role is played by the **Schwartz space**, the space of infinitely differentiable functions that vanish at infinity faster than the reciprocal of any polynomial.

Before defining it, we need to introduce some notational conventions, particularly the use of multi-indices.

Multi-Index Notation

Let us fix a dimension $d \in \mathbb{N}$.

> Multi-Index
>
> A **multi-index** is a vector of non-negative integers:
>
> $$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d) \in \mathbb{N}^d$$
>
> which is used to keep track of the order of differentiation or powers in multiple variable.

The **length** of a multi-index is defined as:

$$|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_d$$

Given a point $x = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d$, we use the shorthand notation:

$$x^\alpha = x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdot \ldots \cdot x_d^{\alpha_d}$$

This notation greatly simplifies expressions involving products of power and derivatives across several variables.

Smooth Functions

We denote by:

$$C^\infty\left(\mathbb{R}^d\right) = \left\{ f : \mathbb{R}^d \to \mathbb{C} \,|\, \partial^\beta f \text{ exists and is continuous } \forall \beta \in \mathbb{N}^d \right\}$$

the space of **smooth (infinitely differentiable) functions** on $\mathbb{R}^d$.
These functions are sufficiently regular to admit derivatives of every order, with each derivative continuous over the whole domain.

1.1 The Schwartz Space

Schwartz Space

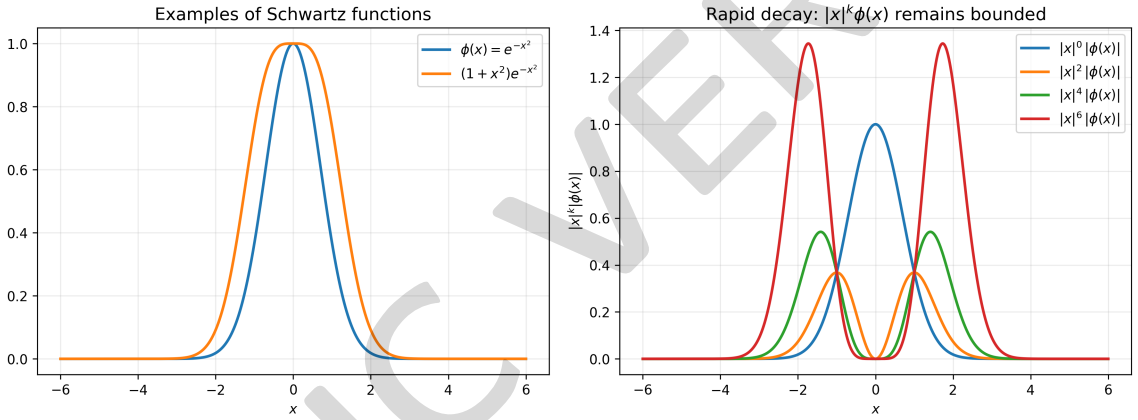
The **Schwartz Space**, denoted by $\mathcal{S}\left(\mathbb{R}^d\right)$, consists of smooth functions that, together with all their derivatives, decay at infinity faster than any polynomial growth. More formally:

$$\mathcal{S}\left(\mathbb{R}^d\right) = \left\{ f \in C^\infty\left(\mathbb{R}^d\right) \mid \forall \alpha, \beta \in \mathbb{N}^d, \sup_{x \in \mathbb{R}^d} \left| x^\alpha \partial^\beta f(x) \right| < +\infty \right\}$$

The condition:

$$\sup_{x \in \mathbb{R}^d} \left| x^\alpha \partial^\beta f(x) \right| < +\infty$$

means that for every derivative $\partial^\beta f$, multiplying it by any polynomial x^α still produces a bounded function.
In other words, derivatives of f vanish at infinity faster than the reciprocal of any polynomial. This ensures that Schwartz functions are "rapidly decreasing".



It is straightforward to check that $\mathcal{S}\left(\mathbb{R}^d\right)$ is a vector space.
If $f, g \in \mathcal{S}\left(\mathbb{R}^d\right)$ and $\lambda \in \mathbb{C}$, then:

$$(f + \lambda g)(x) = f(x) + \lambda g(x) \in \mathcal{S}\left(\mathbb{R}^d\right)$$

Example 1.1 – (Gaussian Functions)

A fundamental example of a Schwartz function is the Gaussian:

$$f(x) = e^{-a|x|^2} \quad , \quad a > 0$$

where $|x|^2 = x_1^2 + \ldots + x_d^2$.
This function, together with all its derivatives, decays to zero faster than any polynomial, making it prototypical element of $\mathcal{S}\left(\mathbb{R}^d\right)$.

1.1.1 Compact Support and Smooth Functions with Compact Support

> Support
>
> The **support** of a function f is defined as:
>
> $$\operatorname{supp}(f) = \overline{\{x \in \mathbb{R}^d \mid f(x) \neq 0\}}$$
>
> namely, the closure of the set where f does not vanish.

A function has **compact support** if this set is bounded and closed, i.e. contained in some large ball of $\mathbb{R}^d$.

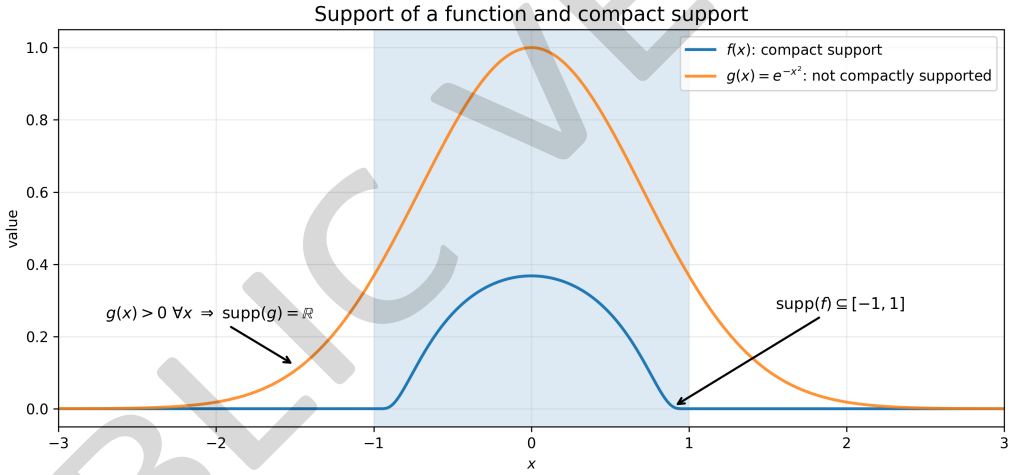
The space of smooth functions with compact support is denoted by:

$$C_0^\infty(\mathbb{R}^d) = \left\{ f \in C^\infty\left(\mathbb{R}^d\right) \mid \operatorname{supp}(f) \text{ is compact} \right\}$$

It is an important fact that:

$$C_0^\infty\left(\mathbb{R}^d\right) \subset \mathcal{S}\left(\mathbb{R}^d\right)$$

because a smooth function that vanishes outside a bounded region is certainly rapidly decreasing (in fact, it vanishes identically beyond a compact set).



1.1.2 Integrability and Lebesgue Spaces

Another important class of function spaces is given by the **Lebesgue spaces** $L^p\left(\mathbb{R}^d\right)$, which consist of measurable functions whose p-th power is integrable.
Specifically, for $1 \leqslant p < \infty$, we define:

$$L^p\left(\mathbb{R}^d\right) = \left\{ f : \mathbb{R}^d \to \mathbb{C} \mid \|f\|_p < +\infty \right\}$$

where:

$$\|f\|_p = \left(\int_{\mathbb{R}^d} |f(x)|^p \, \mathrm{d}x \right)^{1/p}$$

For $p = \infty$, we obtain the space of **essentially bounded functions**, with norm:

$$\|f\|_\infty = \sup_{x \in \mathbb{R}^d} |f(x)|$$

Every Schwartz function belongs to all the Lebesgue spaces simultaneously:

$$\mathcal{S}\left(\mathbb{R}^d\right) \subset L^p\left(\mathbb{R}^d\right) \qquad , \quad \forall\, 1 \leqslant p \leqslant \infty$$

This integrability property makes $\mathcal{S}\left(\mathbb{R}^d\right)$ the ideal setting for Fourier analysis, since both the function and its Fourier transform remain in the Schwartz space.

1.1.3 Embeddings of the Schwartz Space into Lebesgue Spaces

We have already seen that every Schwartz function is integrable in the Lebesgue sense.
A natural question is how to quantify this inclusion, and in particular, whether the embedding of $\mathcal{S}\left(\mathbb{R}^d\right)$ into $L^p\left(\mathbb{R}^d\right)$ is continuous.

Lemma 1.1 ▸ Continuous Embedding

For every $1 \leqslant p \leqslant \infty$, we have the continuous embedding:

$$\mathcal{S}\left(\mathbb{R}^d\right) \hookrightarrow L^p\left(\mathbb{R}^d\right)$$

Here, the arrow $\hookrightarrow$ denotes a continuous injection: not only is every Schwartz function an L^p function, but its L^p norm can be controlled by suitable Schwartz semi-norms.

Proof

Let $f \in \mathcal{S}\left(\mathbb{R}^d\right)$.
For any integer $\delta > 0$, we can write:

$$|f(x)| = \frac{\left(1+|x|^2\right)^{\delta}}{\left(1+|x|^2\right)^{\delta}} \cdot |f(x)|$$

Thus:

$$\|f\|_p^p = \int_{\mathbb{R}^d} |f(x)|^p \, \mathrm{d}x = \int_{\mathbb{R}^d} \left(1+|x|^2\right)^{-\delta p} \left|\left(1+|x|^2\right)^{\delta} \cdot f(x)\right|^p \, \mathrm{d}x$$

We now separate the bounded and integrable parts:

$$\|f\|_p^p \leqslant \sup_{x\in\mathbb{R}^d} \left|\left(1+|x|^2\right)^{\delta} f(x)\right|^p \cdot \int_{\mathbb{R}^d} \left(1+|x|^2\right)^{-\delta p} \mathrm{d}x$$

The first factor is finite because $f \in \mathcal{S}\left(\mathbb{R}^d\right)$ implies rapid decay.
For the second factor, we need to ensure that the integral:

$$I = \int_{\mathbb{R}^d} \left(1+|x|^2\right)^{-\delta p} \mathrm{d}x$$

converges.

Switching to polar coordinates, where $\mathrm{d}x = \rho^{d-1}\,\mathrm{d}\rho\mathrm{d}\omega$ with $\mathrm{d}\omega$ the measure on the unit sphere, we obtain:

$$I = \left|S^{d-1}\right| \int_0^{\infty} \frac{\rho^{d-1}}{(1+\rho^2)^{\delta p}} \, \mathrm{d}\rho$$

Part II

The Theory of Fourier Transform

The Fourier transform is one of the most fundamental tools in mathematical analysis and applied sciences.
It allows us to represent functions as superpositions of oscillatory components, translating problems expressed in the spatial or temporal domain into equivalent ones in the frequency domain.

Many problems in analysis, differential equations, and physics involve understanding how a function behaves in terms of its oscillations or frequencies.
For example, any periodic signal can be decomposed into sines and cosines of different frequencies (via Fourier series).
The Fourier transform extends this idea to **non-periodic** functions defined on all of $\mathbb{R}^d$, allowing a continuous superposition of frequencies.

1 The Fourier Transform

> Intuitively, for a function $f(x)$, its Fourier transform $\hat{f}(\xi)$ measures the "amount" of oscillation at frequency ξ.

The inverse Fourier transform allows one to reconstruct f from its spectral content.

1.1 Definition on the Schwartz Space

We begin by defining the Fourier transform for functions in the Schwartz Space:

$$\mathcal{S}(\mathbb{R}^d) = \left\{ f \in C^\infty\left(\mathbb{R}^d\right) \,:\, \left| x^\alpha D^\beta f(x) \right| \text{ is bounded for all multi-indices } \alpha, \beta \right\}$$

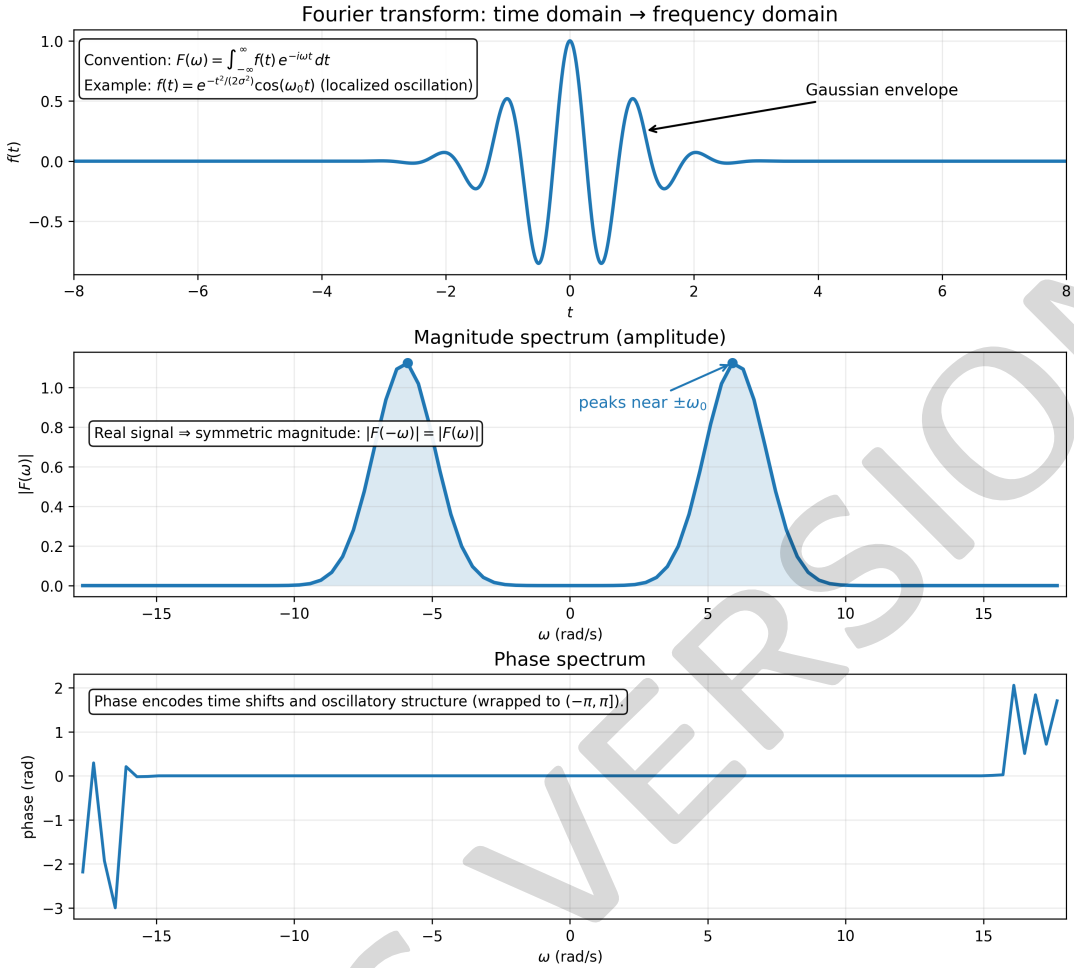
Functions in $\mathcal{S}\left(\mathbb{R}^d\right)$ are infinitely differentiable and decay faster than any polynomial at infinity, making them ideal for Fourier analysis.

Fourier Transform on $\mathcal{S}\left(\mathbb{R}^d\right)$

For $f \in \mathcal{S}\left(\mathbb{R}^d\right)$, the **Fourier transform** is defined by:

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x \cdot \xi} \, \mathrm{d}x$$

where $x \cdot \xi = \sum_{j=1}^{d} x_j \xi_j$ is the Euclidean inner product.



Example 1.1 – (Gaussian Function)

Gaussian functions play a central role in Fourier analysis because they maintain their shape under the Fourier transform. This property makes them fundamental in signal processing, probability theory, and differential equations.

For a parameter $a > 0$, define the **Gaussian function** on $\mathbb{R}^d$ by:

$$\mathcal{G}_a(x) = \exp\left(-\frac{a}{2}\,|x|^2\right) \quad , \qquad x \in \mathbb{R}^d$$

We compute its Fourier transform using the engineering convention:

$$\hat{\mathcal{G}}_a(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i \xi \cdot x} \cdot e^{-\frac{a}{2}|x|^2}\, \mathrm{d}x$$

The exponent contains a quadratic term in x plus a linear oscillatory term.
The standard method is to **complete the square**.

We want to rewrite:

$$-\frac{a}{2}\,|x|^2 - 2\pi i \xi \cdot x$$

as a shifted quadratic plus a correction term. Rewriting the linear term gives

$$-2\pi i \xi \cdot x = -\sqrt{a}x \cdot \frac{2\pi i}{\sqrt{a}} \cdot \xi$$

Thus,

$$-\frac{a}{2}|x|^2 - 2\pi i\xi \cdot x = -\frac{1}{2}\left|\sqrt{a}x + \frac{2\pi i}{\sqrt{a}} \cdot \xi\right|^2 - \frac{2\pi^2}{a}|\xi|^2$$

The idea is that expanding the square gives:

$$\left|\sqrt{a}x + \frac{2\pi i}{\sqrt{a}} \cdot \xi\right|^2 = a|x|^2 + 4\pi i\xi \cdot x - \frac{4\pi^2}{a}|\xi|^2$$

so multiplying by $-1/2$ produces precisely our original terms except for the extra:

$$-\left(-\frac{1}{2}\right) \cdot \frac{4\pi^2}{a}|\xi|^2 = -\frac{2\pi^2}{a}|\xi|^2$$

which must be subtracted out separately.
Thus the Fourier transform becomes

$$\hat{\mathcal{G}}_a(\xi) = \exp\left(-\frac{2\pi^2}{a}|\xi|^2\right) \cdot \int_{\mathbb{R}^d} \exp\left(-\frac{1}{2}\left|\sqrt{a}x + \frac{2\pi i}{\sqrt{a}} \cdot \xi\right|^2\right) \mathrm{d}x$$

Introducing the shift:

$$y = x + \frac{2\pi i}{a} \cdot \xi$$

Multiplying both sides by $\sqrt{a}$,

$$\sqrt{a}y = \sqrt{a}x + \frac{2\pi i}{\sqrt{a}} \cdot \xi$$

Since the shift is constant and imaginary, the Jacobian remains:

$$\mathrm{d}x = \mathrm{d}y$$

Thus the integral reduces to:

$$\int_{\mathbb{R}^d} \exp\left(-\frac{1}{2}\left|\sqrt{a}y\right|^2\right) \mathrm{d}y = \int_{\mathbb{R}^d} \exp\left(-\frac{a}{2}|y|^2\right) \mathrm{d}y$$

Next performing the scaling:

$$\zeta = \sqrt{\frac{a}{2}} \cdot y \qquad \Rightarrow \qquad \mathrm{d}y = \left(\frac{2}{a}\right)^{d/2} \mathrm{d}\zeta$$

Hence,

$$\int_{\mathbb{R}^d} \exp\left(-\frac{a}{2}|y|^2\right) \mathrm{d}y = \left(\frac{2}{a}\right)^{d/2} \cdot \int_{\mathbb{R}^d} e^{-|\zeta|^2} \mathrm{d}\zeta$$

The remaining integral factorizes:

$$\int_{\mathbb{R}^d} e^{-|\zeta|^2} \mathrm{d}\zeta = \left(\int_{\mathbb{R}} e^{-t^2} \mathrm{d}t\right)^d$$

To compute the one-dimensional Gaussian integral, we use polar coordinates in $\mathbb{R}^2$:

1. Compute
$$I = \int_{\mathbb{R}} e^{-t^2} \, \mathrm{d}t$$
2. Then:
$$I^2 = \int_{\mathbb{R}^2} e^{-(x^2+y^2)} \, \mathrm{d}x\mathrm{d}y$$
3. Switch to polar coordinates:
$$I^2 = \int_0^{2\pi} \int_0^{\infty} e^{-r^2} r \, \mathrm{d}r\mathrm{d}\theta = 2\pi \int_0^{\infty} r \cdot e^{-r^2} \mathrm{d}r$$
4. Substitute $u = r^2$, $\mathrm{d}u = 2r \, \mathrm{d}r$:
$$I^2 = \pi \int_0^{\infty} e^{-u} \, \mathrm{d}u = \pi$$
thus $I = \sqrt{\pi}$, therefore:
$$\int_{\mathbb{R}^d} e^{-|\zeta|^2} \, \mathrm{d}\zeta = \left(\sqrt{\pi}\right)^d$$
Combining all factors:
$$\hat{\mathcal{G}}_a(\xi) = \exp\left(-\frac{2\pi^2}{a}|\xi|^2\right)\left(\frac{2}{a}\right)^{d/2}\left(\sqrt{\pi}\right)^d = \left(\frac{2\pi}{a}\right)^{d/2} \cdot \exp\left(-\frac{2\pi^2}{a}|\xi|^2\right)$$
This shows that the Fourier transform of a Gaussian is a Gaussian, but with variance scaled inversely with respect to a.
A Gaussian with parameter a has width proportional to $1/\sqrt{a}$:
- Large $a \to$ function is sharply peaked $\to$ narrow support
- The Fourier transform then contains the factor $\exp\left(-\frac{2\pi^2}{a}|\xi|^2\right)$, whose width is proportional to $\sqrt{a}$
- Therefore, a narrow function in the spatial domain becomes a wide function in the frequency domain

This expresses the classical uncertainty principle in Fourier analysis: **localization in one domain implies spreading in the other**.

1.2 Fundamental Properties

The Fourier transform converts differentiation into multiplication, shifts into modulations, and convolutions into pointwise products.
Throughout we assume
$$f, g \in \mathcal{S}(\mathbb{R}^d)$$
the Schwartz space of rapidly decreasing smooth functions.
In this space all operations (differentiation, integration by parts, exchanging limits) are valid, making it the natural environment for rigorous Fourier analysis.

We also use the convention:

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} \exp\left(-2\pi i \xi \cdot x\right) f(x)\, \mathrm{d}x$$

1.2.1 The Six Core Identities

Let $\alpha, \beta \in \mathbb{N}^d$ be multi-indices.

I) Differentiation ↔ Multiplication

The Fourier transform turns multiplication by x^α into differentiation in ξ:

$$\left(\frac{i}{2\pi}\partial_\xi\right)^\alpha \hat{f}(\xi) = \widehat{(x^\alpha f)}(\xi)$$

Since differentiation can be applied repeatedly, $\hat{f}$ is automatically **smooth** (C^∞) when $f \in \mathcal{S}$.

Proof

Compute the derivative in the first coordinate:

$$\partial_{\xi_1} \hat{f}(\xi) = \partial_{\xi_1} \int_{\mathbb{R}^d} \exp\left(-2\pi i \xi \cdot x\right) f(x)\, \mathrm{d}x$$

Since $f \in \mathcal{S}$, differentiation passes under the integral:

$$= \int_{\mathbb{R}^d} \partial_{\xi_1} \exp\left(-2\pi i \xi \cdot x\right) f(x)\, \mathrm{d}x$$

Using the chain rule:

$$\partial_{\xi_1} \exp\left(-2\pi i \xi \cdot x\right) = -2\pi i x_1 \cdot \exp\left(-2\pi i \xi \cdot x\right)$$

Thus:

$$\partial_{x_1} \hat{f}(\xi) = -2\pi i \int_{\mathbb{R}^d} x_1 f(x) \exp\left(-2\pi i \xi \cdot x\right)\, \mathrm{d}x = -2\pi i \widehat{x_1 f}(\xi)$$

Hence:

$$\left(\frac{i}{2\pi}\partial_{\xi_1}\right) \hat{f}(\xi) = \widehat{x_1 f}(\xi)$$

Repeating in all coordinates yields this first property.

II) Multiplication ↔ Differentiation

Conversely, multiplying the Fourier transform by ξ^β corresponds to differentiating the original function:

$$\xi^\beta \hat{f}(\xi) = \widehat{\left[\left(\frac{1}{2\pi i} \cdot \partial_x\right)^\beta f\right]}(\xi)$$

This identity arises from integration by parts.

5 Hyperbolic PDE: Wave Equation

The wave equation is the prototypical example of a hyperbolic PDE.
Unlike parabolic equations (such as the heat equation seen at Section 3), hyperbolic equations model **propagation phenomena with finite speed**, such as vibrations of strings, sound waves, or electromagnetic waves.

In one spatial dimension, the theory is particularly transparent and admits an explicit solution formula. However, this simplicity is highly dimension-dependent: the structure of solutions differs substantially between dimension 1, odd dimensions (3, 5, 7,...), and even dimensions (2, 4, 6,...). In higher dimensions, the method used here does not directly generalize.

We consider the **homogeneous wave equation** on the real line:

$$\begin{cases} \partial_t^2 u - c^2 \partial_x^2 u = 0 \quad x \in \mathbb{R}\ ,\ t > 0 \\ u(x,0) = g(x) \\ \partial_t u(x,0) = h(x) \end{cases}$$

The equation is second order both in time and space; consequently, two initial conditions are required:

- $g(x)$, representing the initial displacement of the string
- $h(x)$, representing the initial velocity

The equation is called *homogeneous* because the right-hand side is zero, meaning that no external forcing term is present.

To ensure that all derivatives appearing in the equation are well-defined, we assume

$$g \in \mathcal{C}^2(\mathbb{R}) \qquad , \quad h \in \mathcal{C}^1(\mathbb{R})$$

Under these hypotheses, the solution satisfies

$$u \in \mathcal{C}^2(\mathbb{R} \times [0, +\infty))$$

5.1 D'Alembert's Method

The explicit solution of the one-dimensional wave equation is obtained via **D'Alembert's method**, which relies on a change to *characteristic variables*.
We introduce

$$\xi = x + ct \qquad , \qquad \eta = x - ct$$

where the constant $c > 0$ represents the propagation speed of the wave.
Using the chain rule, derivatives transform as follows:

$$\partial_x = \partial_\xi + \partial_\eta \qquad , \qquad \partial_t = c\left(\partial_\xi - \partial_\eta\right)$$

Squaring these operators yields:

$$\partial_x^2 = \partial_\xi^2 + \partial_\eta^2 + 2\partial_{\xi\eta} \qquad , \qquad \partial_t^2 = c^2\left(\partial_\xi^2 + \partial_\eta^2 - 2\partial_{\xi\eta}\right)$$

Substituting into the wave equation gives:

$$\partial_t^2 u - c^2 \partial_x^2 u = -4c^2 \partial_{\xi\eta} u = 0$$

which is equivalent to

$$\partial_{\xi\eta} u = 0$$

This equation is fully equivalent to the original one because the change of variables is smooth and invertible.

The condition $\partial_{\xi\eta} u = 0$ implies that $\partial_\eta u$ does not depend on ξ, hence

$$\partial_\eta u = f_1(\eta)$$

for some function f_1. Integrating with respect to η,

$$u(\xi, \eta) = f_2(\eta) - f_3(\xi)$$

where the integration "constant" may depend on ξ.
Rewriting in the original variables, any solution of the wave equation can be expressed as

$$u(x,t) = \varphi(x + ct) + \psi(x - ct)$$

for suitable functions φ and ψ.
The arguments $x + ct$ and $x - ct$ represent **waves travelling without deformation**:

- $\varphi(x + ct)$ propagates to the left
- $\psi(x - ct)$ propagates to the right

both with speed c.

Imposing the initial conditions yields

$$\varphi(x) + \psi(x) = g(x)$$

and

$$c\varphi'(x) - c\psi'(x) = h(x)$$

Differentiating the first equation and combining with the second gives:

$$\begin{cases} \varphi' + \psi' = g' \\ \varphi' - \psi' = \dfrac{h}{c} \end{cases}$$

Solving,

$$\varphi'(x) = \frac{1}{2}\left[g'(x) + \frac{h(x)}{c}\right] \qquad , \qquad \psi'(x) = \frac{1}{2}\left(g'(x) - \frac{h(x)}{c}\right)$$

Integrating,

$$\varphi(x) = \frac{1}{2}g(x) + \int_0^x \frac{h(s)}{2c}\,\mathrm{d}s + C_1$$
$$\psi(x) = \frac{1}{2}g(x) + \int_0^x \frac{h(s)}{2c}\,\mathrm{d}s + C_2$$

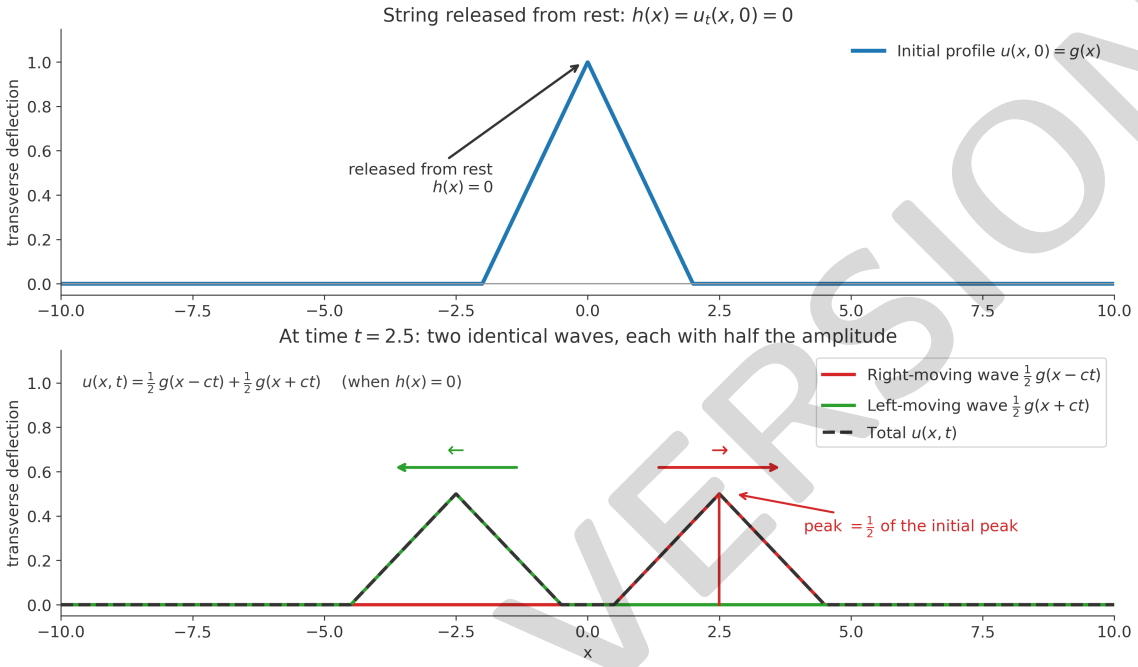
The constants C_1 and C_2 cancel in the final expression and may be set to zero without loss of generality.
Substituting into the general solution and simplifying,

$$u(x,t) = \frac{g(x+ct) + g(x-ct)}{2} + \frac{1}{2c} \cdot \int_{x-ct}^{x+ct} h(s)\,\mathrm{d}s$$

This is known as **D'Alembert's formula**.

Note

Consider a very long string under tension.
The function $g(x)$ represents the initial shape of the string, while $h(x)$ represents its initial velocity.
If the string is displaced at rest and then released, the initial profile splits into two identical waves, each carrying half of the amplitude (half to the right, half to the left).



Theorem 5.1 ▸ Existence and Uniqueness

Let $f \in \mathcal{C}^2(\mathbb{R})$ and $h \in \mathcal{C}^1(\mathbb{R})$.
Then there exists a **unique solution**

$$u \in \mathcal{C}^2(\mathbb{R} \times [0, +\infty))$$

of the wave equation, given by D'Alembert's formula.

Uniqueness follows from the reversibility of every step in the derivation.

Note

If the equation is considered for $t > \bar{t}$, the solution remains valid after replacing

$$t \mapsto t - \bar{t} \qquad , \qquad t - \bar{t} > 0$$

This reflects the **time-translation invariance** of the wave equation.

5.2 Finite Speed of Propagation and Domain of Dependence

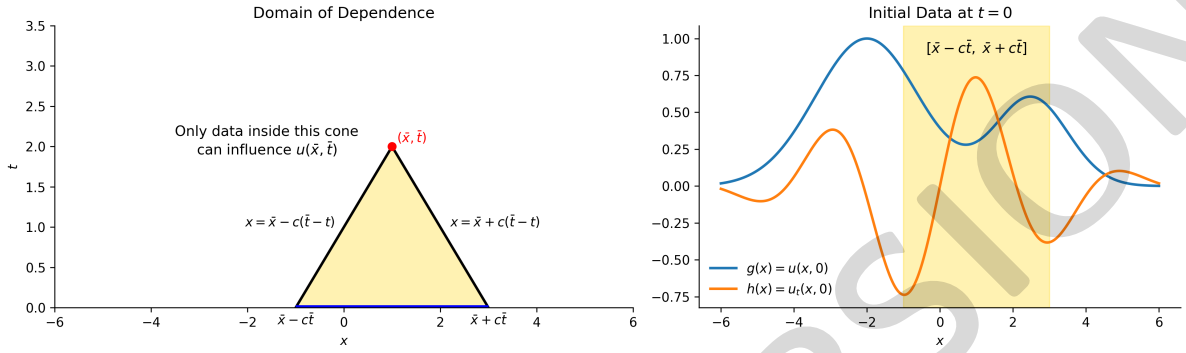
A fundamental property of the wave equation is that information propagates at **finite speed** c. For a given point $(\bar{x}, \bar{t})$, the value $u(\bar{x}, \bar{t})$ depends only on the initial data in the interval

$$[\bar{x} - c\bar{t}, \bar{x} + c\bar{t}]$$

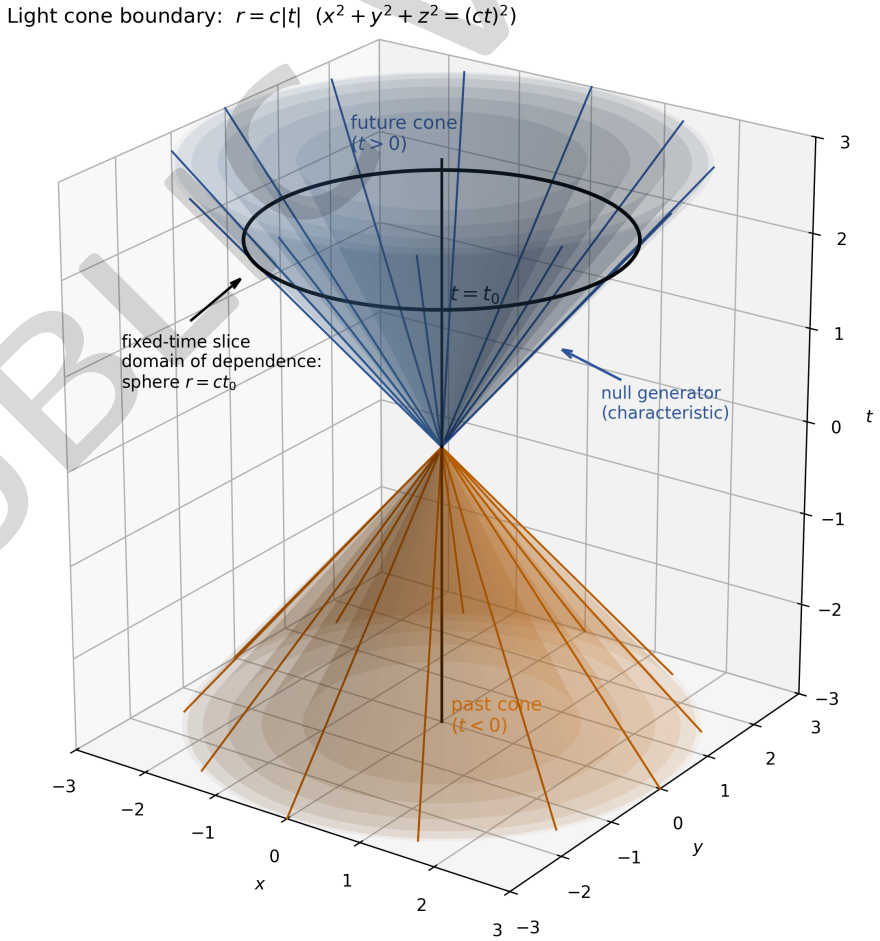
Indeed,

$$u(\bar{x},\bar{t}) = \frac{g(\bar{x}+c\bar{t}) + g(\bar{x}-c\bar{t})}{2} + \frac{1}{2c}\cdot\int_{\bar{x}-c\bar{t}}^{\bar{x}+c\bar{t}} h(s)\,\mathrm{d}s$$

This region is called the **domain of dependence** and is often visualized in the (x,t)-plane as a triangle, or light cone.



In higher dimensions, the same principle holds, but the geometry changes.
In two dimensions, the domain of dependence becomes a **cone**, and the solution depends on values of the initial data on a **circle**.
This behaviour generalizes to higher dimension, though explicit formulas require different techniques.



5.3 The Non-Homogeneous One-Dimensional Wave Equation

We now consider the **forced (non-homogeneous) wave equation**, in which an external source term acts on the system and may depend on both space and time:

$$\begin{cases} \partial_t^2 u - c^2 \partial_x^2 u = f(x,t) & x \in \mathbb{R},\, t > 0 \\ u(x,0) = g(x) & x \in \mathbb{R} \\ \partial_t u(x,0) = h(x) & x \in \mathbb{R} \end{cases}$$

Here, $f(x,t)$ represents an **applied forcing** (for example, a distributed load on a vibrating string).
The PDE remains linear, and this is the key structural property used to solve it.

Linearity and Superposition Principle

Because the operator

$$L[u] \coloneqq \partial_t^2 u - c^2 \partial_x^2 u$$

is linear, we have

$$L\left[u_1 + u_2\right] = L[u_1] + L[u_2]$$

Therefore, we can decompose the solution into two parts,

$$u = u_1 + u_2$$

where:

1. u_1 solves the homogeneous problem with the given initial data:

$$\begin{cases} \partial_t^2 u_1 - c^2 \partial_x^2 u_1 = 0 \\ u_1(x,0) = g(x) \\ \partial_t u_1(x,0) = h(x) \end{cases}$$

   so u_1 is given by D'Alembert's formula derived previously.

2. u_2 solves the forced problem with zero initial data:

$$\begin{cases} \partial_t^2 u_2 - c^2 \partial_x^2 u_2 = f(x,t) \\ u_2(x,0) = 0 \\ \partial_t u_2(x,0) = 0 \end{cases}$$

Then $u = u_1 + u_2$ solves the general system, and uniqueness of the wave equation implies that this decomposition identifies the solution.

From now on we focus on the core new part: solving the zero-data forced problem, i.e. determining u_2. For simplicity, we rename it u.
We study:

$$\begin{cases} \partial_t^2 u - c^2 \partial_x^2 u = f(x,t) \\ u(x,0) = 0 \\ \partial_t u(x,0) = 0 \end{cases}$$